

On solvability and integrability of the Rabi model

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Quasi-exactly solvable Rabi model is investigated within the framework of the Bargmann Hilbert space of analytic functions \mathcal{B} . On applying the theory of orthogonal polynomials, the eigenvalue equation and eigenfunctions are shown to be determined in terms of three systems of monic orthogonal polynomials. The formal Schweber quantization criterion for an energy variable x , originally expressed in terms of infinite continued fractions, can be recast in terms of a meromorphic function $F(z) = a_0 + \sum_{k=1}^{\infty} \mathcal{M}_k/(z - \xi_k)$ in the complex plane \mathbb{C} with *real simple* poles ξ_k and *positive* residues \mathcal{M}_k . The zeros of $F(x)$ on the real axis determine the spectrum of the Rabi model. One obtains at once that, on the real axis, (i) $F(x)$ monotonically decreases from $+\infty$ to $-\infty$ between any two of its subsequent poles ξ_k and ξ_{k+1} , (ii) there is exactly one zero of $F(x)$ for $x \in (\xi_k, \xi_{k+1})$, and (iii) the spectrum corresponding to the zeros of $F(x)$ is necessarily *nondegenerate* in each of the two parity eigenspaces \mathcal{B}_{\pm} . Thereby the calculation of spectra is greatly facilitated. Our results allow us to critically examine recent claims regarding solvability and integrability of the Rabi model.

I. INTRODUCTION

The Rabi model [1] describes the simplest interaction between a cavity mode with a bare frequency ω and a two-level system with a bare resonance frequency ω_0 . The model is characterized by the Hamiltonian [1–3]

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + \hbar g\sigma_1(\hat{a}^\dagger + \hat{a}) + \mu\sigma_3, \quad (1)$$

where \hat{a} and \hat{a}^\dagger are the conventional boson annihilation and creation operators satisfying commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, g is a coupling constant, and $\mu = \hbar\omega_0/2$. In what follows we assume the standard representation of the Pauli matrices σ_j and set the reduced Planck constant $\hbar = 1$. For dimensionless coupling strength $\kappa = g/\omega \lesssim 10^{-2}$, the physics of the Rabi model is well captured by the analytically solvable approximate Jaynes and Cummings (JC) model [4, 5]. The latter is obtained from the former upon applying the rotating wave approximation (RWA), whereby the coupling term $\sigma_1(\hat{a}^\dagger + \hat{a})$ in Eq. (1) is replaced by $(\sigma_+\hat{a} + \sigma_-\hat{a}^\dagger)$, where $\sigma_{\pm} \equiv (\sigma_1 \pm i\sigma_2)/2$. Nowadays, solid-state semiconductor [6] and superconductor systems [7–9] have allowed the advent of the *ultrastrong* coupling regime, where the dimensionless coupling strength $\kappa \gtrsim 0.1$ [10]. In this regime, the validity of the RWA breaks down and the relevant physics can only be described by the full Rabi model [1]. With new experiments rapidly approaching the limit of the *deep strong* coupling regime characterized in that $\kappa \gtrsim 1$ [11], i.e., an order of magnitude stronger coupling, the relevance of the Rabi model [1] becomes even more prominent. There is every reason to believe that ultrastrong and deep strong coupling systems could open up a rich vein of research on truly quantum effects with implications for quantum information science and fundamental quantum optics [6].

The Rabi model applies to a great variety of physical systems, including cavity and circuit quantum electrodynamics, quantum dots, polaronic physics and trapped ions. In spite of recent claims [3, 12], the model is *not* exactly solvable. Rather it is a typical example of *quasi-*

exactly solvable (QES) models in quantum mechanics [13–17]. The QES models are distinguished by the fact that a *finite* number of their eigenvalues and corresponding eigenfunctions can be determined algebraically [13–16]. That is also the case of the Rabi model [17]. Certain energy levels of the Rabi model, known as Juddian exact isolated solutions [18], can be analytically computed [18–20], whereas the remaining part of the spectrum not [19, 20]. Depending on model parameters, the spectrum can only be approximated (sometime rather accurately - cf. Eqs. (18), (20) and Fig. 3 of Ref. [21]; Eq. (20) and Figs. 1,2 of Ref. [22]). Therefore, any kind of exact results involving the Rabi model continues to be of great theoretical and experimental value.

In our earlier work [23] we studied the Rabi model as a member of a more general class \mathcal{R} of quantum models. Models of the class \mathcal{R} were characterized in that the eigenvalue equation

$$\hat{H}\Phi = E\Phi, \quad (2)$$

where \hat{H} denotes a corresponding Hamiltonian, reduces in the Bargmann Hilbert space of analytic functions \mathcal{B} [2, 24] to a *three-term difference equation*

$$\phi_{n+1} + a_n\phi_n + b_n\phi_{n-1} = 0 \quad (n \geq 0). \quad (3)$$

Here $\{\phi_n\}_{n=0}^{\infty}$ are the sought expansion coefficients of a physical state described by an entire function

$$\Phi(z) = \sum_{n=0}^{\infty} \phi_n z^n. \quad (4)$$

Models of the class \mathcal{R} were then characterized in that additionally the recurrence coefficients have an asymptotic powerlike dependence [23]

$$a_n \sim an^{\varsigma}, \quad b_n \sim bn^{\upsilon} \quad (n \rightarrow \infty), \quad (5)$$

where a and b are proportionality constants and the exponents satisfy $2\varsigma > \upsilon$ and $\tau = \varsigma - \upsilon \geq 1/2$ [23]. The

spectrum of any quantum model of \mathcal{R} can be obtained as zeros of the transcendental function of a dimensionless energy parameter x [23, 25],

$$F(x) \equiv a_0 + \sum_{k=1}^{\infty} \rho_1 \rho_2 \dots \rho_k, \quad (6)$$

where $F(x)$ is defined solely in terms of the coefficients of the three-term recurrence [23, 25, 26]:

$$\begin{aligned} \rho_1 &= -\frac{b_1}{a_1}, \quad \rho_l = u_l - 1, \quad u_1 = 1, \\ u_l &= \frac{1}{1 - u_{l-1}b_l/(a_l a_{l-1})}, \quad l \geq 2. \end{aligned} \quad (7)$$

The function $F(x)$ yields both analytic and efficient numerical representation of the formal Schweber quantization criterion expressed in terms of infinite continued fractions (cf. Eq. (A.16) of Ref. [2]),

$$F(x) \equiv a_0 + \frac{-b_1}{a_1 - \frac{b_2}{a_2 - \frac{b_3}{a_3 - \dots}}} \quad (8)$$

The Rabi Hamiltonian \hat{H}_R is known to possess a discrete \mathbb{Z}_2 -symmetry corresponding to the constant of motion, or parity, $\hat{\Pi} = \exp(i\pi\hat{J})$ [3, 11, 19], where

$$\hat{J} = \hat{a}^\dagger \hat{a} + \frac{1}{2} (1 + \sigma_3) \quad (9)$$

is the familiar operator known to generate a continuous $U(1)$ symmetry of the JC model [3, 4]. The Rabi model can be characterized by a pair of the three-term recurrences (Eq. (37) of Ref. [23])

$$\begin{aligned} \phi_{n+1}^\pm + \frac{1}{\kappa(n+1)} [n - \epsilon \pm (-1)^n \Delta] \phi_n^\pm \\ + \frac{1}{n+1} \phi_{n-1}^\pm = 0. \end{aligned} \quad (10)$$

Here $\epsilon \equiv E^\pm/\omega$, $\kappa = g/\omega$ reflects the coupling strength, $\Delta = \mu/\omega$, and \pm applies to the *positive* and *negative* parity states, respectively, with power series expansions $\Phi^\pm(z) = \sum_{n=0}^{\infty} \phi_n^\pm z^n$ for the respective parity states being assumed [23]. The Bargmann Hilbert space of analytic functions can be thus written as a direct sum $\mathcal{B} = \mathcal{B}_+ \oplus \mathcal{B}_-$ of the parity eigenspaces. The case of a displaced harmonic oscillator, which is the exactly solvable limit of \hat{H} for $\mu = 0$, results in $\Delta = 0$, whereby the recurrence (10) reduces to Eq. (A.17) of Ref. [2].

In the case of the Rabi model, and its special case of the displaced harmonic oscillator, it was observed that the plots of $F(\epsilon)$ corresponding to (10) displayed a series of *discontinuous* branches *monotonically decreasing* between $+\infty$ and $-\infty$ (see Fig. 1 and Figs. 1,2 of Ref. [23]). In the present work the latter property will be proven. First we show that $F(x)$, considered as a function of $x \equiv \epsilon/\kappa = E^\pm/g$, can be alternatively expressed

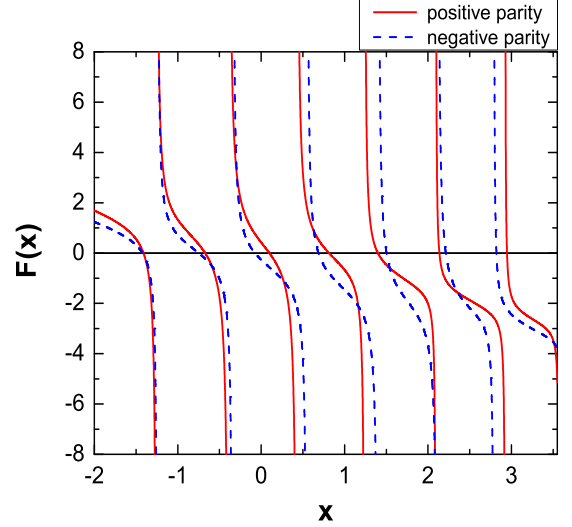


Figure 1. $F(x)$ corresponding to the recurrence (10) in the deep strong coupling regime for $\kappa = 1.4$, $\Delta = 0.4$, $\omega = 1$, and $x = \epsilon/\kappa = E/g$.

as the limit of rational functions

$$F(x) \equiv \lim_{n \rightarrow \infty} F_n(x) = a_0 + \lim_{n \rightarrow \infty} \frac{P_{n-1}^{(1)}(x)}{P_n(x)}, \quad (11)$$

where $\{P_n(x)\}_{n=0}^{\infty}$ and $\{P_n^{(1)}(x)\}_{n=0}^{\infty}$ are associated systems of *monic orthogonal polynomials*. (Monic means here that the coefficient of the highest power of x is one.) The polynomials of each orthogonal polynomial system (OPS) $\{P_n(x)\}_{n=1}^{\infty}$ and $\{P_n^{(1)}(x)\}_{n=1}^{\infty}$

- have *real* and *simple* zeros, and
- the zeros of $P_{n-1}^{(1)}(x)$ and $P_n(x)$ are *interlaced*.

Specifically, denote the zeros of $P_n(x)$ with degree $P_n = n$ by $x_{n1} < x_{n2} < \dots < x_{nn}$ and the zeros of $P_{n-1}^{(1)}(x)$ with degree $n-1$ by $x_{n-1,1}^{(1)} < x_{n-1,2}^{(1)} < \dots < x_{n-1,n-1}^{(1)}$. Then for any $k = 1, 2, \dots, n-1$

$$x_{n,k}^{(\alpha)} < x_{n-1,k}^{(\alpha)} < x_{n,k+1}^{(\alpha)}, \quad (12)$$

$$x_{nk} < x_{n-1,k}^{(1)} < x_{n,k+1}, \quad (13)$$

where $\alpha = 0, 1$ (for the sake of notation the superscript (0) for $\alpha = 0$ will be suppressed in what follows). For each fixed k , $\{x_{nk}\}_{n=k}^{\infty}$ is a *decreasing* sequence and the limit

$$\xi_k = \lim_{n \rightarrow \infty} x_{nk} \quad (14)$$

exists. Additionally, for any finite n the ratio in (11), also known as a *convergent*, enables a partial fraction

decomposition (PFD)

$$\frac{P_{n-1}^{(1)}(x)}{P_n(x)} = \sum_{k=1}^n \frac{M_{nk}}{x - x_{nk}}. \quad (15)$$

The numbers M_{nk} are all *positive*, $M_{nk} > 0$, and satisfy the condition

$$\sum_{k=1}^n M_{nk} = 1. \quad (16)$$

Each number M_{nk} can be shown to correspond to the weight corresponding to the zero x_{nk} in the Gauss quadrature formula for the positive definite moment functional \mathcal{L} associated with the OPS $\{P_n(x)\}_{n=0}^\infty$. In the case of the displaced harmonic oscillator and the Rabi model,

$$\begin{aligned} M_{nk} &= -\frac{(n+1)!}{P_{n+1}(x_{nk})P'_n(x_{nk})} \\ &= \left(\sum_{l=0}^{n-1} \frac{P_l^2(x_{nk})}{(l+1)!} \right)^{-1} > 0. \end{aligned} \quad (17)$$

From (15) one finds immediately that whenever the derivative $dF_n(x)/dx$ exists, then

$$\frac{dF_n(x)}{dx} < 0. \quad (18)$$

Consequently, between any two subsequent $x_{nk} < x_{n,k+1}$, where $F_n(x)$ decreases from $+\infty$ to $-\infty$, there is exactly one zero of $F_n(x)$, in agreement with Fig. 1 and Figs. 1,2 of Ref. [23]. Hence, $F_n(x)$ has its zeros and poles interlaced on the real axis. That would also conclude any numerical method of computing $F(x)$ through Eq. (11), because of an unavoidable cutoff at some $n = N \gg 1$.

The above conclusions remain valid also in the limit $n \rightarrow \infty$. A point of crucial importance is that the inequality (12) survives the limit as the sharp inequality

$$\xi_k < \xi_{k+1} \quad (19)$$

for all $k \geq 1$. The sequence in Eq. (11) converges to a *Mittag-Leffler* PFD,

$$F(z) = a_0 + \sum_{k=1}^{\infty} \frac{\mathcal{M}_k}{z - \xi_k}, \quad (20)$$

defining a meromorphic function in the complex plane \mathbb{C} with *real simple* poles and *positive* residues

$$0 < \mathcal{M}_k = \left[\sum_{l=0}^{\infty} \frac{P_l^2(\xi_k)}{(l+1)!} \right]^{-1}. \quad (21)$$

The series is absolutely and uniformly convergent in any finite domain having a finite distance from the simple poles ξ_j , and it defines there a holomorphic function of z

[$z \in \mathbb{C}$ here and below has no relation to z in Eq. (4)]. One obtains at once that

$$\frac{dF(x)}{dx} < 0. \quad (22)$$

Because $F(x)$ monotonically decreases from $+\infty$ to $-\infty$ between any its subsequent poles ξ_k and ξ_{k+1} , there is exactly one zero of $F(x)$ for $x \in (\xi_k, \xi_{k+1})$. As a byproduct, the spectrum in each parity eigenspace \mathcal{B}_\pm is necessarily *nondegenerate*. Eventually, the knowledge of another OPS, $\{P_n^{(-1)}(x)\}_{n=1}^\infty$, enables one to determine the expansion coefficients of a physical state described by (4) as

$$\phi_n(x) = \frac{P_n^{(-1)}(x)}{n!}. \quad (23)$$

We prove the above results in the forthcoming section II, which is divided into two subsections. First, subsection II A deal with the case of an arbitrary large but finite $n = N$. The limit $N \rightarrow \infty$ is then considered in subsection II B. Section III illustrates some of our findings on the exactly solvable case of the displaced harmonic oscillator. In Sec. IV our results are then extensively discussed from various angles. Subsection IV A gives a comparison of the properties of our F with those of Braak's functions G_\pm . In subsections IV B and IV C recent claims regarding solvability and integrability of the Rabi model [3, 12] are critically examined. Compatibility of our results with some other results of the theory of infinite continued fractions and complex analysis is demonstrated in subsection IV D. Subsection IV E gives an overview of some open problems. We then conclude with Sec. V. Some additional technical remarks are relegated to Appendix A.

II. PROOF OF THE MAIN RESULTS

According to the Wallis formulas (Eqs. (III.2.1) of Ref. [28]; Eqs. (4.2-3) of Ref. [29]), given a three-term recurrence (3), the infinite continued fraction in Eq. (8) can be expressed as the limit

$$r_0 = \lim_{n \rightarrow \infty} \frac{A_n}{B_n}, \quad (24)$$

where the n th *partial numerator* A_n and the n th *partial denominator* B_n are determined as linearly independent solutions of the recurrence

$$A_n = a_n A_{n-1} - b_n A_{n-2}, \quad (25)$$

$$B_n = a_n B_{n-1} - b_n B_{n-2}, \quad (26)$$

where $n \geq 1$. The A_n 's and B_n 's are differentiated by the initial conditions:

$$A_{-1} = 1, \quad A_0 = 0, \quad B_{-1} = 0, \quad B_0 = 1. \quad (27)$$

In an intriguing and peculiar world of infinite continued fractions, the respective recurrences (25) and (26) are essentially identical to the initial three-term recurrence (3) (up to the change $a_n \rightarrow -a_n$ and the omission of the $n = 0$ term). For the Rabi model we have (Eq. (37) of Ref. [23], or Eq. (10) herein above)

$$a_n = -\frac{1}{(n+1)} \left(\frac{\epsilon}{\kappa} - \bar{c}_n \right), \quad b_n = \frac{1}{n+1}, \quad (28)$$

$$\bar{c}_n \equiv \frac{1}{\kappa} [n \pm (-1)^n \Delta]. \quad (29)$$

A. Arbitrary large but finite N

In order to prove the properties of $F_n(x)$ defined by Eq. (11) for an arbitrary n , together with the properties listed below, it is sufficient to prove that each of the recurrences (10), (25), (26) can be transformed into a recurrence of the type

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \quad (30)$$

$$P_{-1}(x) = 0, \quad P_0(x) = 1, \quad (31)$$

where $n \geq 1$, the coefficients c_n and λ_n are *real* and independent of x , and $\lambda_n > 0$. Obviously, the above recurrence defines a family of polynomials $\{P_n\}_{n=0}^\infty$ with degree $P_n = n$. According to Favard-Shohat-Natanson theorems (given as Theorems I-4.1 and I-4.4 of Ref. [28]), satisfying the above recurrence is a necessary and sufficient condition that there exists a unique positive definite moment functional \mathcal{L} , such that for the family of polynomials $\{P_n\}$ holds

$$\mathcal{L}[1] = \lambda_1, \quad \mathcal{L}[P_m(x)P_n(x)] = \lambda_1 \lambda_2 \dots \lambda_{n+1} \delta_{mn}, \quad (32)$$

$m, n = 0, 1, 2, \dots$ and δ_{mn} is the Kronecker symbol. Thereby the polynomials $\{P_n\}$ form an OPS [30]. Because $\lambda_n > 0$, the norm of the polynomials P_n is positive definite, $\mathcal{L}[P_n^2(x)] > 0$, and \mathcal{L} is *positive definite* moment functional (p. 16 of Ref. [28]).

With a_n and b_n as in Eqs. (28), a substitution $B_n = (-1)^n P_n / (n+1)!$ transforms the three-term recurrence (26) into the recurrence of the type (30) and (31) with

$$c_n = \bar{c}_n, \quad \lambda_n = \bar{\lambda}_n \equiv n, \quad (33)$$

where \bar{c}_n has been defined by Eq. (29) and $x = \epsilon/\kappa = E/g$. A similar substitution $A_n = (-1)^n S_n / (n+1)!$ transforms (25) into the recurrence

$$S_n(x) = (x - \bar{c}_n)S_{n-1} - \lambda_n S_{n-2} \quad (34)$$

with $\lambda_n = n$, but with a “wrong” initial condition

$$S_{-1} = -1, \quad S_0 = 0. \quad (35)$$

The latter is not of the required type (31). Note that a

recurrence of the type (34) yields

$$\begin{aligned} S_1 &= \lambda_1, \\ S_2 &= (x - \bar{c}_2)\lambda_1, \\ S_3 &= (x - \bar{c}_3)(x - \bar{c}_2)\lambda_1 + \lambda_3\lambda_1, \\ S_4 &= (x - \bar{c}_4)(x - \bar{c}_3)(x - \bar{c}_2)\lambda_1 + (x - \bar{c}_4)\lambda_3\lambda_1 \\ &\quad + (x - \bar{c}_2)\lambda_4\lambda_1, \\ &\dots \end{aligned} \quad (36)$$

Therefore, a further substitution $S_n = \lambda_1 Q_{n-1}$ transforms the recurrence (34) into

$$Q_n(x) = (x - \bar{c}_{n+1})Q_{n-1}(x) - \lambda_{n+1}Q_{n-2}(x), \quad (37)$$

where $n \geq 1$, with the “correct” initial conditions

$$Q_{-1} = 0, \quad Q_0 = 1. \quad (38)$$

The recurrence (37) together with the initial conditions is now of the type (30) and (31) with

$$c_n = \bar{c}_{n+1}, \quad \lambda_n = \bar{\lambda}_{n+1} = n+1. \quad (39)$$

Eventually, a substitution $\phi_n \rightarrow \bar{\phi}_n / n!$, transforms the initial recurrence (10) into

$$\bar{\phi}_{n+1}^\pm = (x - \bar{c}_n)\bar{\phi}_n^\pm - \bar{\lambda}_n \bar{\phi}_{n-1}^\pm. \quad (40)$$

The recurrence for $\bar{\phi}_n^\pm$ is again of the type (30) and (31) with

$$c_n = \bar{c}_{n-1}, \quad \lambda_n = \bar{\lambda}_{n-1} = n-1, \quad (41)$$

where we set $\lambda_1 = \bar{\lambda}_0 = 1 \neq 0$ for $n = 1$. Note in passing that λ_1 enters the recurrence (30) only in the product $\lambda_1 P_{-1}$, where P_{-1} satisfies the initial condition (31). Therefore we have the freedom to set λ_1 at our will. The respective recurrences for $\bar{\phi}_n$'s, P_n 's, and Q_n 's have all been shown to be of the type

$$P_n^{(\alpha)}(x) = (x - \bar{c}_{n+\alpha})P_{n-1}^{(\alpha)}(x) - \bar{\lambda}_{n+\alpha}P_{n-2}^{(\alpha)}(x), \quad (42)$$

$$P_{-1}^{(\alpha)}(x) = 0, \quad P_0^{(\alpha)}(x) = 1, \quad (43)$$

where the coefficients \bar{c}_n and $\bar{\lambda}_{n+\alpha}$ are *real* and independent of x , and $\bar{\lambda}_{n+\alpha} > 0$ for $n \geq 1$. One has $\alpha = -1, 0, 1$ for $\bar{\phi}_n$'s, P_n 's, and Q_n 's, respectively.

We continue to denote the polynomials of the OPS for $\alpha = 0$ by $\{P_n\}_{n=0}^\infty$. They determine the denominators B_n 's in Eq. (24). The respective monic OPS with $\alpha = -1, 1$ are called *associated* to P_n 's and will be denoted by $\{P_n^{(\alpha)}\}_{n=0}^\infty$ (see Sec. III-4 of Ref. [28]). Because

$$A_n = \frac{(-1)^n P_{n-1}^{(1)}}{(n+1)!}, \quad B_n = \frac{(-1)^n P_n}{(n+1)!}, \quad (44)$$

it follows at once that the ratio (24) can be expressed as the limit of the ratios of the orthogonal monic polynomi-

als

$$r_0 = \lim_{n \rightarrow \infty} \frac{P_{n-1}^{(1)}(x)}{P_n(x)}. \quad (45)$$

The properties listed below Eq. (11) follow straightforwardly from the classic theory of orthogonal polynomials (see esp. Secs. I.4-6 and III.1-4 of Ref. [28]). The zeros of the polynomials of any OPS are *real* and *simple* (Theorem I-5.2 of Ref. [28]). Furthermore, the zeros of any two subsequent polynomials $P_n(x)$ and $P_{n+1}(x)$ of an OPS mutually separate each other (Theorem I-5.3 of Ref. [28]). The separation property of zeros (13) follows from Theorem III-4.1 of Ref. [28]. Eqs. (14) and (16) follow from Eqs. (I-5.6) and (I-6.2) of Ref. [28], where we have assumed $\mathcal{L}[1] \equiv \mu_0 = 1$. The partial fraction decomposition (15) follows from Theorem III-4.3 of Ref. [28]. The positivity of M_{nk} in Eq. (17) follows from the Christoffel-Darboux identity (Eq. (I-4.13) of [28]),

$$P'_{n+1}(x)P_n(x) - P'_n(x)P_{n+1}(x) > 0, \quad (46)$$

which for $x = x_{nk}$ reduces to

$$P_{n+1}(x_{nk})P'_n(x_{nk}) < 0, \quad (47)$$

where the prime denotes derivatives. The 2nd of Eqs. (17) follows from Theorem I-4.6 of Ref. [28]. Thereby our results for any finite $n = N$ have been proved.

Note in passing that the coefficients M_{nk} 's in the PFD (15) satisfy, in virtue of the Gauss quadrature formula (Eq. (II-3.1) of Ref. [28]),

$$\sum_{k=1}^n M_{nk} x_{nk}^l = \mu_l \quad (l = 0, 1, 2, \dots, 2n-1), \quad (48)$$

where μ_l 's are the corresponding moments of the positive definite moment functional, $\mathcal{L}[x^l] = \mu_l$. (The positivity of \mathcal{L} implies $\mu_{2l} > 0$, but not necessarily $\mu_{2l+1} > 0$.)

B. The limit $N \rightarrow \infty$

According to the representation theorem (Theorem II-3.1 of Ref. [28]), the weight function ψ of the positive moment functional \mathcal{L} (also called *distribution* function [28]),

$$\mathcal{L}[x^n] = \int_{-\infty}^{\infty} x^n d\psi(x) = \mu_n \quad (n = 0, 1, \dots), \quad (49)$$

is the limit of a sequence of bounded, right continuous, nondecreasing step functions $\psi_n(x)$'s,

$$\begin{aligned} \psi_n(x) &= 0 & (-\infty \leq x < x_{n1}), \\ \psi_n(x) &= M_{n1} + \dots + M_{np} & (x_{np} \leq x < x_{n,p+1}), \\ \psi_n(x) &= \mu_0 & (x \geq x_{nn}). \end{aligned} \quad (50)$$

Consequently

- $\psi_n(x)$ has exactly n points of increase, x_{nk} ,
- the discontinuity of $\psi_n(x)$ at each x_{nk} equals M_{nk} ($k = 1, 2, \dots, n$),
- at least the first $(2n-1)$ moments of the weight function $\psi_n(x)$ are identical with those of $\psi(x)$, i.e.,

$$\int_{-\infty}^{\infty} x^l d\psi_n(x) = \mu_l \quad (l = 0, 1, 2, \dots, 2n-1). \quad (51)$$

Obviously, for any $z \in \mathbb{C}$ different from the zeros x_{nk} 's the PFD in Eq. (15) can be expressed as

$$\frac{P_{n-1}^{(1)}(z)}{P_n(z)} = \sum_{k=1}^n \frac{M_{nk}}{z - x_{nk}} = \int_{-\infty}^{\infty} \frac{d\psi_n(x)}{z - x}. \quad (52)$$

According to Hamburger's Theorem XII' [31], the function

$$f(z) = \int_{-\infty}^{\infty} \frac{d\psi(x)}{z - x}, \quad (53)$$

where the Stieltjes integral measure $d\psi$ has been defined through the limit of $\psi_n(x)$'s, is a *regular analytic* function in any closed finite region Ω of the complex plane which does not contain any part of the real axis. The convergents in Eq. (52) converge *uniformly* to $f(z)$ in Ω . According to Definition III-1.1 of [28], the infinite continued fraction in Eqs. (8) and (24) then converges and

$$F(z) = a_0 + \int_{-\infty}^{\infty} \frac{d\psi(x)}{z - x}. \quad (54)$$

So far we have mostly summarized the relevant classical results of Hamburger [31]. A point of crucial importance in our case is that the resulting Stieltjes measure $d\psi(x) \equiv \psi(x) - \psi(x-0)$ is necessarily *discrete*. (Here $\psi(x-0)$ denotes the left-side limit of ψ at x , $\psi(x-0) = \lim_{x_n \rightarrow x, x_n < x} \psi(x_n)$.) To this end, we first show that the set of zeros x_{nk} extends beyond any bound up to at $+\infty$. Denote

$$\sigma \equiv \lim_{j \rightarrow \infty} \xi_j, \quad (55)$$

where ξ_j 's are the limit zero points defined by Eq. (14). According to Eq. (IV-3.7) of Ref. [28], a sufficient condition for $\sigma = \infty$ is that

$$\lim_{n \rightarrow \infty} c_n = \infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{\lambda_{n+1}}{c_n c_{n+1}} < \frac{1}{4}. \quad (56)$$

In the present case of the Rabi model, with \bar{c}_n and $\bar{\lambda}_n$ defined by Eq. (33), one has $\bar{\lambda}_{n+1}/(\bar{c}_n \bar{c}_{n+1}) = \mathcal{O}(n^{-1})$. The conditions (56) are then obviously satisfied,

$$\lim_{n \rightarrow \infty} \bar{c}_n = \infty, \quad \limsup_{n \rightarrow \infty} \frac{\bar{\lambda}_{n+1}}{\bar{c}_n \bar{c}_{n+1}} = 0. \quad (57)$$

Now the condition $\sigma = \infty$ ensures that the limit zero points ξ_k defined by (14) are all *distinct*, i.e., Eq. (19) holds. Indeed, if $\xi_k = \xi_{k+1}$ for some k , then ξ_k is a limit point of ξ_l 's (Theorem II-4.4 of Ref. [28]). According to Theorem II-4.6 of Ref. [28], if $\xi_k = \xi_{k+1}$ for some $k \geq 1$, then

$$\xi_k = \sigma \equiv \lim_{j \rightarrow \infty} \xi_j. \quad (58)$$

(Such a separation of the limit zero points $\xi_k^{(\alpha)}$'s applies also to the other two OPS with $\alpha \neq 0$.) Because of the sharp inequality (19), the Stieltjes weight function ψ satisfies $\psi = \text{const}$ on any interval $x \in (\xi_k, \xi_{k+1})$. Consequently $d\psi$ is a *discrete* measure. Moreover, the measure $d\psi$ is unambiguously determined (see footnote 50 on p. 268 of Ref. [31]). The determinacy of $d\psi$, and the Stieltjes weight function ψ (assuming the normalization $\psi(-\infty) = 0$), follows also independently from Carleman's criterion which says that the moment problem is determined if (cf. Eq. (VI-1.14) of Ref. [28])

$$\sum_{l=1}^{\infty} \lambda_l^{-1/2} = \infty. \quad (59)$$

The latter is obviously satisfied in our case.

Now the support of the measure induced by ψ , or briefly the spectrum of ψ , is precisely the set of all ξ_k 's (pp. 63 and 113 of Ref. [28]). Therefore, by the very definition of the spectrum of ψ (p. 51 of Ref. [28]), the residues in (54) are strictly positive, $0 < d\psi(\xi_k) = \mathcal{M}_k$, which proves Eq. (21) for $z \in \Omega$ having a finite distance $\delta > 0$ from the real axis. As the result, the Stieltjes integrals in Eqs. (53) and (54) reduce to infinite sums.

In what follows we show by the Stieltjes-Vitali theorem (cf. p. 121 of Ref. [28]; p. 144 of Ref. [32]) that the convergents in Eq. (52) converge *uniformly* to a *regular analytic* function $f(z)$ represented by Eq. (53) not only in any closed finite $\Omega \cap \mathbb{R} = \emptyset$, such as in Hamburger's Theorem XII' [31], but also on the real axis for $z = x \in (\xi_k + 2\delta, \xi_{k+1} - \delta)$, where $\delta > 0$ is sufficiently small real number (see Fig. 2). To this end we remind that the sum over all M_{nk} 's satisfies Eq. (16). Therefore (cf. Eq. (56) of Ref. [32]),

$$\left| \frac{P_{n-1}^{(1)}(z)}{P_n(z)} \right| \leq \sum_{k=1}^n \frac{M_{nk}}{|z - x_{nk}|} \leq \frac{1}{\delta}, \quad (60)$$

where $\delta = \min |z - x_{nk}|$. Now select any pair of subsequent limit zero points ξ_k and ξ_{k+1} . Because the limit zero points are separated, there exists some infinitesimal $\delta > 0$ so that $\xi_k + 2\delta < \xi_{k+1} - \delta$. Now consider $z = x \in (\xi_k + 2\delta, \xi_{k+1} - \delta)$. Because each sequence x_{nk} is decreasing with increasing n , $x_{nk} \notin (\xi_k + \delta, \xi_{k+1})$ for sufficiently large $n \geq N$. Then $\min |x - x_{nk}| \geq \delta > 0$. This establishes a uniform bound on the convergents for $z = x \in (\xi_k + 2\delta, \xi_{k+1} - \delta)$. The latter bound can be obviously extended to any closed region in the complex

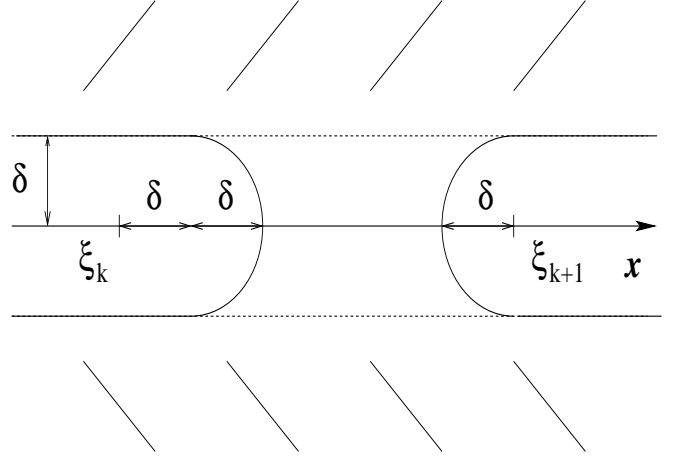


Figure 2. Analyticity domain of $F(z)$ guaranteed by Hamburger's theorem [31] extends to the half-planes above and below the real axis at the distance of at least δ from the latter as showed by the dashed lines. In the case of the Rabi model there is possible to extend the domain of analyticity of $F(z)$ through the real axis in any interval (ξ_k, ξ_{k+1}) and to connect the upper and lower half-planes.

plane bounded with a semicircle of radius δ at $\xi_k + \delta$ and a semicircle of radius δ at $\xi_{k+1} - \delta$ and having a distance at least δ from the real axis for $\text{Re } z \in (-\infty, \xi_k + \delta]$ and $\text{Re } z \in [\xi_{k+1} - \delta, \infty)$ (see Fig. 2). Analyticity is then established by the Stieltjes-Vitali theorem [28, 32]. Thus $F(z)$ takes on the form of the *Mittag-Leffler* PFD (20), which is absolutely and uniformly convergent in any finite domain having a finite distance from the simple poles ξ_k 's.

III. EXAMPLE OF THE DISPLACED HARMONIC OSCILLATOR

The recurrence (72) has for $\Delta = 0$, i.e., in the case of the displaced harmonic oscillator, the solution [2]

$$\phi_n = \kappa^{-n} L_n^{(\zeta-n)}(\kappa^2), \quad (61)$$

where $\zeta = \epsilon + \kappa^2$ and L_n^η are associated Laguerre polynomials [33] (note different sign of κ compared to Eq. (2.16) of Schweber [2]). Therefore,

$$P_n^{(-1)}(x) = n! \kappa^{-n} L_n^{(\kappa x + \kappa^2 - n)}(\kappa^2). \quad (62)$$

Let instead of the recurrence (30) polynomials Q_n satisfy

$$Q_n(x) = \left(x - \frac{c_n - q}{p} \right) Q_{n-1}(x) - \frac{\lambda_n}{p^2} Q_{n-2}(x). \quad (63)$$

Then

$$Q_n(x) = p^{-n} P_n(px + q) \quad (p \neq 0). \quad (64)$$

Given Eqs. (62), (63), and (64),

$$P_n(x) = P_n^{(-1)}[x - (1/\kappa)] = n! \kappa^{-n} L_n^{(\kappa x - n)}(\kappa^2). \quad (65)$$

The generalized Laguerre polynomial of degree n is

$$L_n^{(\zeta)}(x) = \sum_{j=0}^n (-1)^j \binom{n+\zeta}{n-j} \frac{x^j}{j!}, \quad (66)$$

which implies

$$\phi_n = \sum_{j=0}^n (-1)^j \binom{\zeta}{n-j} \frac{\kappa^{2j-n}}{j!}, \quad (67)$$

where the (generalized) binomial coefficients

$$\binom{\zeta}{k} := \frac{\zeta(\zeta-1)(\zeta-2)\cdots(\zeta-k+1)}{k!}. \quad (68)$$

Eq. (67) shows that ϕ_n is a sum of polynomials in ζ ,

$$\phi_n = \sum_{j=0}^n (-1)^{n-j} \frac{\kappa^{n-2j}}{(n-j)! j!} \prod_{l=0}^j (\zeta - l), \quad (69)$$

where $\zeta = l$ corresponds to the eigenvalue

$$\epsilon_l = l - \kappa^2 \quad (70)$$

of the displaced harmonic oscillator [2]. Each ϕ_n is also a polynomial of degree n in energy parameter x , which can be verified for first few coefficients explicitly:

$$\begin{aligned} \phi_0 &= L_0^{(\zeta)}(\kappa^2) = 1, \\ \phi_1 &= \kappa^{-1} L_1^{(\zeta-1)}(\kappa^2) = \epsilon/\kappa = x, \\ \phi_2 &= \kappa^{-2} \left[\frac{\kappa^4}{2} - \zeta \kappa^2 + \frac{\zeta(\zeta-1)}{2} \right] \\ &= \frac{x}{2!} \left(x - \frac{1}{\kappa} \right) - \frac{1}{2!}, \\ \phi_3 &= \kappa^{-3} \left[-\frac{\kappa^6}{3!} + \frac{\zeta \kappa^4}{2} \right. \\ &\quad \left. - \frac{\zeta(\zeta-1)\kappa^2}{2} + \frac{\zeta(\zeta-1)(\zeta-2)}{6} \right] \\ &= \frac{x}{3!} \left(x - \frac{1}{\kappa} \right) \left(x - \frac{2}{\kappa} \right) - \frac{x}{2!} + \frac{1}{3\kappa}. \end{aligned} \quad (71)$$

Note that $\phi_1/\phi_0 = x$, which is exactly the $n=0$ part of (10).

IV. DISCUSSION

Our recent work has been driven by the curiosity as to what extent the recurrence coefficients a_n and b_n of a model from \mathcal{R} determine the model basic properties [23, 25]. Earlier we looked at the problem from the per-

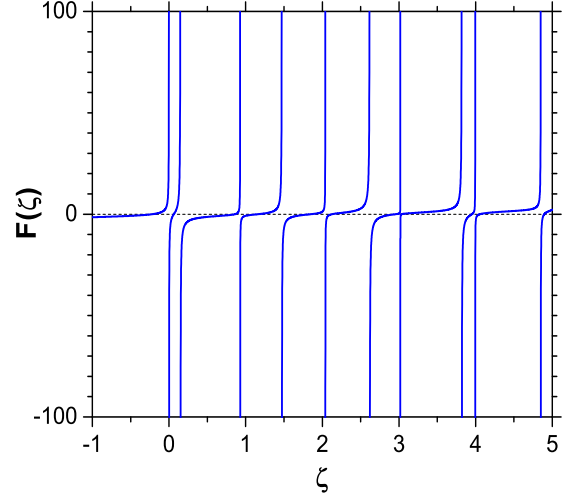


Figure 3. Plot of $F(\zeta)$ corresponding to the recurrence (72) for $g = 0.7$, $\Delta = 0.4$, and $\omega = 1$, i.e. the same parameters as for $G_{\pm}(\zeta)$ in Fig. 1 of Ref. [3], shows corresponding zeros at ≈ -0.217805 , 6.29563×10^{-2} , 0.86095 , 1.1636 , 1.85076 , etc.

spective of the minimal solutions [29] of the recurrence (3) and showed that the spectrum of any quantum model from \mathcal{R} can be obtained as zeros of a transcendental function F [23, 25, 26]. In the present work we took a complementary view and analyzed in detail the analytic structure of Schweber's quantization condition [2]. We showed that the function F originally defined by Eqs. (6) and (7) can alternatively be represented by the *Mittag-Leffler* PFD (20) with repelling zeros and with positive residues \mathcal{M}_k defined by (21). The latter enabled to prove the monotonicity of $F(z)$ and *nondegeneracy* of the spectrum of the Rabi model in each parity eigenspace \mathcal{B}_{\pm} . The non-degeneracy is not new result - it was initially obtained by Kus [19] within the framework of Frobenius's analysis of regular singular points [27]. Yet it is both stimulating and inspiring that our approach based on OPS's enabled us to prove the basic analytic property of the Rabi model independently, without any recourse to its earlier proof.

We have presented our results while treating the special case of the Rabi model. However, as obvious from the proof, our results remain valid for any model of the class \mathcal{R} which can be reduced to the recurrence of the type (30) and (31) and which satisfies the conditions (56). (more general sufficient conditions than (56) can be found in Sec. IV-3 of Ref. [28].) Essential for obtaining our results was to start with the pair of the recently uncovered parity resolved three-term recurrences (10) (cf. Eq. (37) of Ref. [23]). The recurrences are different from the orig-

inal recurrence for the Rabi model [2],

$$\phi_{n+1} - \frac{f_n(\zeta)}{(n+1)}\phi_n + \frac{1}{n+1}\phi_{n-1} = 0, \quad (72)$$

where $f_n(\zeta)$ is given by

$$f_n(\zeta) = 2\kappa + \frac{1}{2\kappa} \left(n - \zeta - \frac{\Delta^2}{n - \zeta} \right), \quad (73)$$

where $\kappa = g/\omega$ and $\Delta = \mu/\omega$ are as in Eq. (10) (cf. Eq. (A8) of Schweber [2], which has mistyped sign in front of his b_{n-1} , and Eqs. (4) and (5) of [3]). The dimensionless energy parameter $\zeta = (E/\omega) + \kappa^2$ is the same as in Sec. III.

Because f_n in (73) contains ζ both in a numerator and in a denominator, the recurrence (72) does not reduce to that of the type (30) and (31) obeyed by OPS's. Nevertheless, as shown in Fig. 3, the transcendental function $F(\zeta) \equiv -f(\zeta) + r_0$ corresponding to (72) still displays a series of *discontinuous* branches *monotonically* extending between $+\infty$ and $-\infty$ and the spectrum can be obtained as zeros of $F(\zeta)$ (cf. Fig. 1 of Ref. [25, 26]). However, one can no longer guarantee that all the residues \mathcal{M}_k are positive nor ensure the sharp inequality (19). Note that a kind of a *Mittag-Leffler* PFD (20) is rather general (cf. Eq. (67) of Grommer [32]). It also applies to OPS's which distribution function has a bounded denumerable spectrum with a finite number of limit points (cf. Theorems 3.2 and 5.4 of Ref. [34]).

A. A comparison with Braak's functions

In a recent letter [3], Braak claimed to have solved the Rabi model analytically (see also Viewpoint by Solano [12]). He suggested [3] that a *regular* spectrum of the Rabi model was given by the zeros of transcendental functions $G_{\pm}(\zeta)$,

$$G_{\pm}(\zeta) = \sum_{n=0}^{\infty} K_n(\zeta) \left[1 \mp \frac{\Delta}{\zeta - n} \right] \kappa^n. \quad (74)$$

Here the coefficients $K_n(\zeta)$ were obtained recursively by solving the Poincaré difference equation (72) upwardly starting from the initial condition

$$\phi_1/\phi_0 = -a_0 = 2\kappa - \frac{1}{2\kappa} \left(\zeta - \frac{\Delta^2}{\zeta} \right). \quad (75)$$

Braak argued that between subsequent poles of the term in the square bracket in (74) at $\zeta = n$ and $\zeta = n+1$ the function $G_{\pm}(\zeta)$ takes on zero value:

- *once* - by implicitly presuming that at one of the poles $G_{\pm}(\zeta)$ goes to $+\infty$ and at the neighboring pole goes to $-\infty$, with a monotonic behavior from $+\infty$ to $-\infty$ between the poles;

- *twice* - implicitly presuming that $G_{\pm}(\zeta)$ goes to one of $\pm\infty$ at both subsequent poles of the term in the square bracket in (74), and in between the poles it has rather *featureless* behavior, e.g., similar to the cord hanging on two posts;
- *none* - occurs under the similar circumstances as described in the previous item, if the “cord is too short”, e.g., it does not stretch sufficiently up or down as to cross the abscissa.

However, the above behavior can be merely regarded as an *unproven* hypothesis. There is no proof in Ref. [3] that the above behavior is the only possible behavior. In this regard, Eqs. (72) and (73) show that $K_n(\zeta)$ is a complicated function given as a sum of polynomial fractions with increasing polynomial degree in both the numerator and denominator. Therefore, each $K_n(\zeta)$ has its own zeros and poles structure. Consequently, it cannot be excluded that between any two subsequent poles of the term in the square bracket in (74) the function $G_{\pm}(\zeta)$ would display much more complicated behavior than that assumed by Braak [3]. The same objections apply to the proposed functional form of $G_{\pm}(\zeta)$ in Eq. (6) of Ref. [3],

$$G_{\pm}(\zeta) = G_{\pm}^0(\zeta) + \sum_{n=0}^{\infty} \frac{h_n^{\pm}}{\zeta - n}, \quad (76)$$

where $G_{\pm}^0(\zeta)$ is entire in ζ . There is no proof in Ref. [3] that $K_n(\zeta) \neq 0$ nor that $K_n(\zeta)$ is free of poles [cf. Eqs. (72) and (73)]. Actually the Mittag-Leffler PFD (20) of our F function is much stronger result than the unproven Eq. (76) of Braak [3]. Whereas Braak [3] cannot say anything about the residues h_n^{\pm} , the residues \mathcal{M}_k in the Mittag-Leffler PFD (20) are all positive and given by Eq. (21). There is also no entire function contribution in our PFD (20). Furthermore, the F function has exactly one zero between any two of its subsequent poles.

B. Solvability

A typical quasi-exactly solvable Schrödinger operator can be expressed as a polynomial of degree at most two in the generators of the $sl(2, \mathbb{R})$ algebra [14, 35],

$$H = \sum_{k,m} q_{km} J_k^n J_m^n + \sum_m q_m J_m^n + q_*, \quad (77)$$

where q_{km} , q_m , q_* are some *real* constants,

$$J_-^n = \partial_z, \quad J_0^n = z\partial_z - \frac{n}{2}, \quad J_+^n = z^2\partial_z - nz, \quad (78)$$

n is an integer, and

$$[J_+^n, J_-^n] = 2J_0^n, \quad [J_0^n, J_{\pm}^n] = \pm J_{\pm}^n. \quad (79)$$

The Rabi model [17] allows for such a representation in terms of the generators of $sl(2, \mathbb{R})$ algebra only for the

energies satisfying Eq. (70) (cf. Eq. (12) of Ref. [17]), which correspond to the eigenvalues of the displaced harmonic oscillator. The lines $l - \kappa^2$ are exactly the baselines on which the Juddian exact isolated solutions occur [18–20]. The latter correspond to the degenerate solutions of the Rabi model [19] and coincide with the intersection points of different parity solutions as a function of energy [19]. The baselines are the only places where such a degeneracy is possible [19]. This shows that the very existence of the Juddian exact isolated solutions is a consequence of that the Rabi model is quasi-exactly solvable. The quasi-exactly solvable eigenstates are exactly the parity *degenerate* Juddian isolated exact solutions [18–20].

However, not any quasi-exactly solvable Schrödinger operator can be expressed in terms of the quadratic elements of an enveloping $sl(2, R)$ algebra [36]. Therefore one might argue that an exact solvability for other energy values would still be possible and the claims of Braak [3] could be justified. Nevertheless, the possibility of polynomial solutions can be excluded by recent result by Zhang [37]. Indeed, let us consider the differential equation

$$\left[X(z) \frac{d^2}{dz^2} + Y(z) \frac{d}{dz} + Z(z) \right] \Phi(z) = 0, \quad (80)$$

where $X(z) = \sum s_k z^k$, $Y(z) = \sum t_k z^k$, $Z(z) = \sum v_k z^k$ are polynomials of degree at most 4, 3, 2, respectively. Zhang [37] found all polynomials $Z(z)$ such that Eq. (80) has polynomial solutions $\Phi(z) = \prod_{j=1}^l (z - z_j)$ of degree l with distinct roots z_j . Theorem 1.1 of Zhang [37] yields an algebraic conditions on each of the coefficients v_k , $k = 0, 1, 2$ of $Z(z)$ in terms of the coefficients $X(z)$, $Y(z)$, and the roots z_j . In the case of the Schweber's equation for the Rabi model [Eq. (3.23) of Ref. [2]], and upon taking into account that Schweber's κ is twice of ours,

$$\begin{aligned} & z(z - 2\kappa) \frac{d^2 \Phi(z)}{dz^2} \\ & + [2(\kappa\zeta - \kappa) + (1 - 2\zeta + 4\kappa^2)z - 2\kappa z^2] \frac{d\Phi(z)}{dz} \\ & + [\zeta^2 - \Delta^2 - 2\kappa\zeta(1 - z)] \Phi(z) = 0. \end{aligned} \quad (81)$$

Because for the Rabi model Zhang's $s_4 = s_3 = t_3 = 0$, Zhang's condition (1.8) on v_2 reduces to $v_2 = 0$, and Zhang's condition (1.9) on v_1 reduces to $v_1 = -lt_2$, or,

$$\zeta = l. \quad (82)$$

The latter leads to (70), i.e., again to the Juddian exact isolated solutions [18–20]. One arrives at the same conclusion if Zhang's conditions [37] are applied to Eq. (5) of [17], which is another variant of (81). Therefore, the Rabi model has no other polynomial solution than the Juddian exact isolated solutions.

A possible signature of exact solutions could be seen in the case of the displaced harmonic oscillator. There the solution of the recurrence for ϕ_n was given by Eq. (69).

In general, rapid growths of the product $\prod_{k=0}^n (\zeta - \zeta_k)$ in (69) essentially cancels out the $1/n!$ prefactor, leads to a *finite* radius of convergence of the series for $\Phi(z)$ in Eq. (4), and prevents $\Phi(z)$ from being an element of \mathcal{B} . The points of the spectra are characterized by a sudden collapse of the degree of ϕ_n , which is in general the polynomial of degree n in energy, to a polynomial of merely the $(l - 1)$ th order for any $n \geq l$ at the l th spectral point (including $l = 0$). Indeed, $\zeta = \epsilon + \kappa^2$ is a nonnegative integer l at the points of the spectra and the sum over j in Eq. (67) runs only between $j = 0$ and $j = l - 1$ for $n \geq l$. Otherwise the product on the right-hand side of Eq. (67) vanishes. The leading $(l - 1)$ th order in ζ is then rapidly decreasing with increasing n as

$$(-1)^{n+1-l} \frac{l\kappa^{n+2-2l}}{(n+1-l)!}, \quad (83)$$

which implies $\Phi \in \mathcal{B}$ [2]. Thus the very same product terms $\prod_{k=0}^j (\zeta - \zeta_k)$, which initially prevented $\Phi(z)$ from being an element of \mathcal{B} , come later on to the rescue and ensure that $\Phi(z) \in \mathcal{B}$ for the spectral points $\zeta = \zeta_l = l$.

It appears plausible that ϕ_n 's can be expressed as a sum of such product terms over the spectrum of a model and the points of the spectra would again be characterized by such a sudden collapse of the degree of ϕ_n also in general case (e.g., for the Rabi model). However, even in the case of the displaced harmonic oscillator, the ϕ_n 's will be generated by the recurrence (40) as polynomials in x . The leading order of ϕ_n in x is then provided by the polynomial term

$$\frac{1}{n!} \prod_{l=0}^{n-1} \left(x - \frac{l}{\kappa} \right). \quad (84)$$

Starting from the recurrence (40), it is highly nontrivial to arrive at Eq. (67) then. The latter is established neither here nor by Braak [3].

C. Integrability

In his work on the Rabi model Braak [3] suggested the following criterion of quantum integrability: If each eigenstate of a quantum system with f_1 discrete and f_2 continuous degrees of freedom (d.o.f.) can be uniquely labeled by $f_1 + f_2 = f$ quantum numbers $\{d_1, \dots, d_{f_1}, c_1, \dots, c_{f_2}\}$, such that the d_j can take on $\dim(\mathcal{H}_j)$ different values, where \mathcal{H}_j is the state space of the j th discrete d.o.f. and the c_k range from 0 to infinity, then this system is *quantum integrable*.

The Rabi model has $f_1 = f_2 = 1$ and all degeneracies have been known to take place only between levels of states with different parity, whereas within the parity subspaces no level crossings occur [19, 20]. Consequently, the spectral graph of the Rabi model consists of two “ladders” of level lines $|p, m\rangle$. Each ladder corresponds to an invariant subspace of the \mathbb{Z}_2 symmetry

characterized by $p = \pm$, the parity eigenvalue [19, 20]. Because the global label (valid for all values of g) of the Rabi model is two dimensional as $f_1 + f_2 = 2$, Braak [3] then concluded that the Rabi model belongs to the class of integrable systems. However, the above reasoning is not conclusive. First, the very notion of quantum integrability is the subject of ongoing dispute [38]. Second, the nearest-neighbor distribution of energy levels is customarily used to distinguish between integrable models and chaotic systems [39–41]. If a normalized spacing between two energy levels is s , then, according to the Berry-Tabor universality conjecture [39, 41], integrable models are generically characterized by a Poisson distribution $\mathcal{P}(s) = e^{-s}$, which peaks up at the zero level spacing, $\mathcal{P}(0) = 1$. On the other hand, according to the Bohigas-Giannoni-Schmit conjecture [40], for chaotic systems is symptomatic a *level repulsion* [40, 41], i.e., $\mathcal{P}(0) = 0$. For instance, for time-invariant systems the normalized distribution of level spacings of chaotic systems is well approximated by the Wigner distribution $\mathcal{P}(s) = (\pi s/2) \exp\{-\pi s^2/4\}$ of eigenvalues of the Gaussian orthogonal ensemble of random matrices [40, 41]. Given our result that the limit zeros points ξ_k repel each other [cf. Eq. (19)], and that the zeros of $F(x)$ are found typically midway between subsequent pairs ξ_k and ξ_{k+1} , one finds support rather for the *level repulsion*, and hence chaotic dynamics of the Rabi model in each of the parity invariant subspaces \mathcal{B}_{\pm} . Obviously, further work is needed to resolve the integrability issue. In this regard, our approach based on the properties of a characteristic function $F(x)$ defined by Eqs. (6) and (7) is well positioned to deal with that task. The reason is that, with the exception of the very first zero, the presence of each subsequent zero of $F(x)$ is announced in advance by a preceding pole of $F(x)$ (see Fig. 1 and Figs. 1,2 of Ref. [23]). This makes the search of all the zeros of $F(x)$, which correspond to the spectrum of the Rabi model, quite straightforward.

D. Compatibility

With increasing n , the PFD (15), (52) define a sequence of rational functions with simple real poles and positive residues. Kritikos (§ 4 of Ref. [42]) showed that if the sequence of such rational functions

$$R_n(z) = \sum_{k=1}^n \frac{M_{nk}}{z - x_{nk}} \quad (85)$$

converges *uniformly* in a proximity of some point $z_0 \in \mathbb{C}$, then the sequence converges everywhere in the complex plane with a possible exception of the real axis. The convergence is *uniform* in any bounded region of the complex plane with a nonzero distance from the real axis. The latter is obviously compatible with our main result.

In general one cannot always guarantee that, such as in Eq. (21), $M_{nk} \rightarrow \mathcal{M}_k = d\psi > 0$ in the limit $n \rightarrow \infty$.

Indeed, Theorem I in §8 of Grommer [32] merely ensures that for any $m > 0$

$$\begin{aligned} \psi_n(x_{nk} - 0) &< \psi_{n+m}(x_{nk}) < \psi_n(x_{nk}) \\ &= \psi_n(x_{n,k+1} - 0) < \psi_{n+m}(x_{n,k+1}), \end{aligned} \quad (86)$$

where as usual $\psi_n(x - 0)$ denotes the left-side limit of ψ_n at x . On taking the limit $n \rightarrow \infty$ one cannot exclude that $\psi(\xi_k) = \psi(\xi_{k+1})$ for some k , and hence $d\psi(\xi_{k+1}) \equiv 0$. The above Grommer's theorem appears to be related to the fact that, given the sharp inequality (19), $\xi_k^{(1)}$ could coincide with one of $\xi_k < \xi_{k+1}$. Because $\xi_k^{(1)} < \xi_{k+1}^{(1)}$, one can only exclude that two subsequent $\xi_k^{(1)}$ and $\xi_{k+1}^{(1)}$ coincide with a single ξ_k or ξ_{k+1} . If $\xi_k^{(1)} = \xi_{k+1}$, then $d\psi(\xi_{k+1}) = 0$ in the *Mittag-Leffler* PFD (20).

On combining the special cases of Eqs. (I-4.12) and (III-4.4) of Ref. [28] for $x = x_{n+1,k}$ and on substituting for $P_n(x_{n+1,k})$ from the former to the latter, one obtains

$$\frac{P_n^{(1)}(x_{n+1,k})}{P'_{n+1}(x_{n+1,k})} = \left[\sum_{l=0}^n \frac{P_l^2(x_{n+1,k})}{(l+1)!} \right]^{-1}. \quad (87)$$

On comparing with the right-hand side of Eq. (21) one finds that at the support of $d\psi$ the left-hand side of (87) has a *nonzero* limit for $n \rightarrow \infty$. That implies the sharp inequality

$$\xi_k < \xi_k^{(1)} < \xi_{k+1}, \quad (88)$$

meaning that the interlacing property of zeros (13) for a finite n survives the limit $n \rightarrow \infty$. Note that the sum such as in Eqs. (21) and (87) enters also the celebrated Chebyshev inequalities (cf. Theorem II-5.5 of Ref. [28]).

E. Open problems

The dynamics and long-time evolution of the Rabi model is well understood only for rather weak couplings $\kappa = g/\omega \lesssim 10^{-2}$, where the Rabi model can be reliably approximated by the JC model [4, 5]. The latter was originally proposed as an exactly-solvable approximation to the Rabi model by applying the RWA and neglecting rapidly oscillating counterrotating terms [4, 5]. The JC model provides the basis for, from the Rabi model perspective somewhat misleadingly called, *strong-coupling* regime [6, 10, 11]. The latter encompasses the cavity quantum electrodynamics and associated with it vacuum-field Rabi oscillations of atoms, molecules, and quantum-dots in a cavity [6]. Long-time behavior of various dynamical variables of the JC model can be described by analytic approximations [43] and the dynamics shows periodic spontaneous collapse and revival of coherence [43, 44]. With new experiments rapidly approaching the limit of the deep strong coupling regime characterized by $\kappa \gtrsim 1$ [11], the question of major physical relevance is that of the dynamics of the Rabi model for $\kappa \gtrsim 0.1$

[11]. A great deal of insight into the dynamics of the Rabi model has been gained by Casanova et al. [11] in the limit $\omega_0/\omega \ll 1$ by means of an expansion in the small parameter ω_0/ω . Photon wave number packets were shown to propagate coherently along two independent parity chains of states and, like in the JC model [43, 44], exhibited a collapse-revival pattern of the system population [11]. Nevertheless, still only very little is known about the dynamics in a very interesting physical region of $\omega_0/\omega \simeq 1$. The just established link between the Rabi model and the OPS's could improve calculations of the eigenvalues and eigenfunctions, which are prerequisite for a reliable calculation of the dynamics of the Rabi model, and shed light on its long-time evolution for all values of the dimensionless coupling κ [11].

Carleman's criterion states that the Hamburger moment problem associated with the positive-definite OPS (30) and (31) is *determined* (i.e., the Stieltjes weight function ψ is unique), if the condition (59) is satisfied [28]. An alternative sufficient condition for the determinacy of the classical Hamburger moment problem is that the moments (49) of the positive moment functional \mathcal{L} have to satisfy (cf. Eq. (3) of Ref. [45]; §5 of Ref. [46])

$$|\mu_n| \leq \frac{\theta}{\Lambda^n} n!, \quad (89)$$

where θ and Λ are two real positive constants. Hamburger's condition (89) is also a sufficient condition for the *uniform* convergence of the associated infinite continued fraction in any *closed* domain Ω of the complex plane which does not contain any part of the real axis [45]. The sequence of convergents in Eqs. (15) and (52) then converges *uniformly* to the infinite continued fraction in Eq. (8), which ensures the convergence of the latter [45] (see also pp. 214-215 of Ref. [46]). This raises the question regarding a mutual relation between the respective asymptotic behaviors of the λ_n 's and μ_n 's. In the special case of $c_n \equiv 0$ in Eq. (30) for all $n \in \mathbb{N}$, Bender and Milton [47] showed that $\lambda_n \sim n$ implies $\mu_n \sim n!$ and vice versa. It would be interesting to prove rigorously if such a relation holds also in more general case of $c_n \neq 0$, such as for the Rabi model.

Last but not the least, an interesting question is if the ideas presented here could also be extended to systems characterized by a three-term difference equation (3) with *periodic* coefficients, like in the Hofstadter problem of Bloch electrons in rational magnetic fields [48].

V. CONCLUSIONS

On applying the theory of orthogonal polynomials, the eigenvalue equation and eigenfunctions of the quasi-exactly solvable Rabi model was shown to be determined in terms of three systems of monic orthogonal polynomials. The formal Schweber quantization criterion for an energy variable x , originally expressed in terms of infi-

nite continued fractions, was shown to be equivalent to a meromorphic function $F(x)$ in the complex plane \mathbb{C} expressed by the partial fraction decomposition (20) with *real simple* poles and positive residues \mathcal{M}_k defined by (21). One obtains at once that (i) $F(x)$ monotonically decreases from $+\infty$ to $-\infty$ between any two of its subsequent poles ξ_k and ξ_{k+1} , (ii) there is exactly one zero of $F(x)$ for $x \in (\xi_k, \xi_{k+1})$, and (iii) the spectrum corresponding to the zeros of $F(x)$ is in each parity eigenspace \mathcal{B}_\pm necessarily *nondegenerate*. Thereby the calculation of spectra was greatly facilitated. Recent claims regarding solvability and integrability of the Rabi model [3, 12] were critically examined (see subsections IV B and IV C). Compatibility of our results with some other results of the theory of infinite continued fractions and complex analysis was explicitly demonstrated.

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Appendix A: Technical remarks

Assuming that the power series $\sum_{n=0}^{\infty} \mu_n/z^n$ has a nonzero radius of convergence $|z| > R > 0$, i.e., much weaker condition than Hamburger's condition (89), Grommer arrived from an integral representation (cf. Eq. (63) of Ref. [32])

$$f(z) = \int_{-R}^R \frac{d\psi(x)}{z-x} \quad (A1)$$

to the *Mittag-Leffler* PFD (cf. Eq. (66) of Ref. [32])

$$f(z) = \sum_{j=1}^{\infty} \frac{\mathcal{M}_j}{z-\xi_j}. \quad (A2)$$

Essential to his arguments was that $f(z)$ remained finite and different from zero for any nonreal $z \in \mathbb{C}$. However, in virtue of

$$\int_{-\infty}^{\infty} \frac{dx}{(u-x)^2 + q^2} = \frac{\pi}{q} < \infty. \quad (A3)$$

the very same is also true if the integration range in Eq. (A1) were, such as in our case, extended to infinity.

By making use of the Nevanlinna theorem [49] (later rediscovered by Sokal [50] in his improvement of Watson's theorem on Borel summability; for an extension of the Nevanlinna-Sokal theorem to differently shaped region see Ref. [51]), which was not known to Hamburger at the time he wrote his [45], one can immediately amend Hamburger theorem under his item 4 on pp. 33-34 of [45] and prove that Hamburger's function $f(z)$ is asymptotically represented by the power series $S(z)$ not merely in

an angular domain $\varepsilon \leq \arg z \leq \pi - \varepsilon$, where ε an arbitrary small positive infinitesimal, but rather in a disc tangent to the real axis.

The zeros of $P_n(x)$ can be associated with eigenvalues of *Jacobi* matrices. The latter are derived from the coefficients of the recurrence (30) upon writing each $\lambda_n > 0$ as a product of two complex conjugate numbers $\lambda_n = z_n z_n^*$, and assembling them into the Hermitian matrix

$$M_n = \begin{bmatrix} c_1 & z_2 & 0 & \dots & 0 & 0 & 0 \\ z_2^* & c_2 & z_3 & \dots & 0 & 0 & 0 \\ 0 & z_3^* & c_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & z_{n-1}^* & c_{n-1} & z_n \\ 0 & 0 & 0 & \dots & 0 & z_n^* & c_n \end{bmatrix}. \quad (\text{A4})$$

Then the eigenvalues of M_n are the zeros of $P_n(x)$. (Ex. I-5.7 on p. 30 of Ref. [28]).

Using the results of Hamburger [45], one can prove that

$$w_n(t) = \sum_{k=1}^n M_{nk} e^{x_{nk}t} \quad (\text{A5})$$

converges uniformly toward

$$W(t) = \sum_{n=0}^{\infty} \frac{\mu_n}{n!} t^n \quad (\text{A6})$$

within any vertical stripe $|t| \leq R - \delta$, where $R > 0$ is a nonzero radius of convergence of the series for $W(t)$ and δ is some infinitesimally small number. The continued fraction in Eqs. (8) and (24) converges then uniformly to the Borel transform

$$F(z) = a_0 + \int_0^{-i\infty} W(t) e^{-tz} dt \quad (\text{A7})$$

in any closed domain of the complex plane with $\text{Im } z \geq \delta > 0$ (cf. § 3 of Ref. [45]).

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$$P_n(x) = (\beta_n x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \quad (\text{A8})$$
where the coefficients β_n , c_n and λ_n are independent of x , $\beta_n \neq 0$, and $\lambda_n \neq 0$ for $n \geq 1$ [28].
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